Von Neumann Stability Analysis

- An initial line of errors (represented by a finite Fourier series) is introduced and the growth or decay of these errors in time (or iteration) dictates stability.
- A finite Fourier series is expressed in the form:

$$\sum_{j=1}^{N} C_j \exp(i\sigma_j x) \qquad \text{in 1-D}$$

where

$$\sigma_j = 2\pi \frac{j}{L} \qquad \text{and} \quad i = \sqrt{-1}$$

The number of terms N is equal to the number of mesh points on the line.

- Assumes linear constant coefficient finite difference approximations, uniform mesh spacing, and boundaries at infinity.
- Linearity permits Fourier components to be treated separately and superposition used to add all other components

Recall: Fourier series of a function f(x) in D[a,b]

$$f(x) = \frac{a_o}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2k\pi x}{(b-a)} \right) + b_k \sin\left(\frac{2k\pi x}{(b-a)} \right) \right]$$

With

$$a_k = \frac{2}{(b-a)} + \int_a^b f(x) \cos\left(\frac{2k\pi x}{(b-a)} \right) dx$$

$$b_k = \frac{2}{(b-a)} + \int_a^b f(x) \sin\left(\frac{2k\pi x}{(b-a)}\right) dx$$

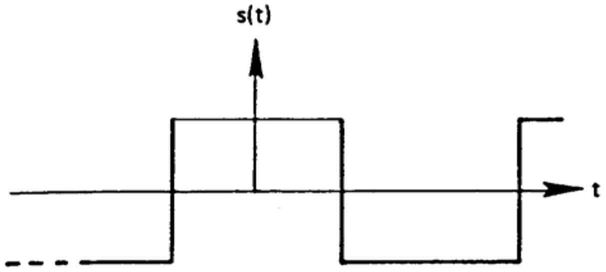
Also,

$\exp(i\theta) = \cos(\theta) + i \sin(\theta)$

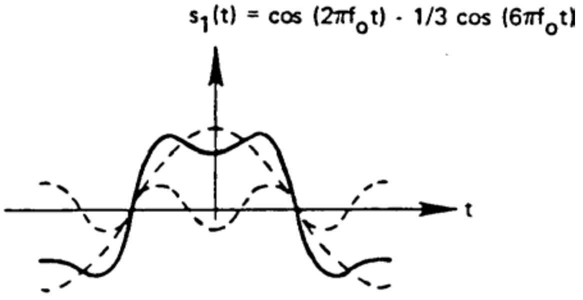
$\cos(\alpha) = [\exp(i\,\alpha) + \exp(-i\,\alpha)]/2$

$\sin(\alpha) = [\exp(i\,\alpha) - \exp(-i\,\alpha)]/(2i)$

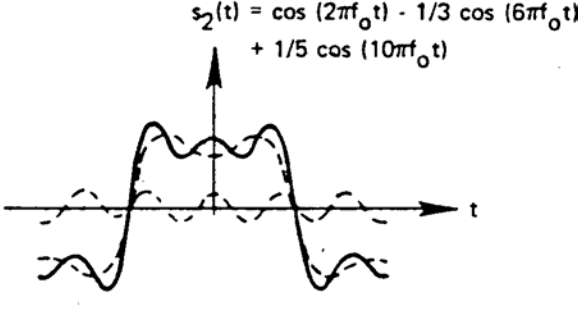
Any waveform can be decomposed into Fourier components. Consider a Square Wave:

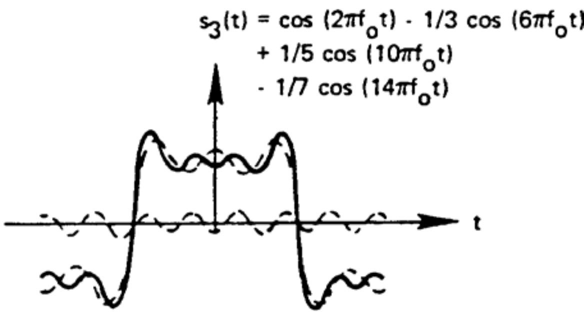


If two cosine terms are added together:



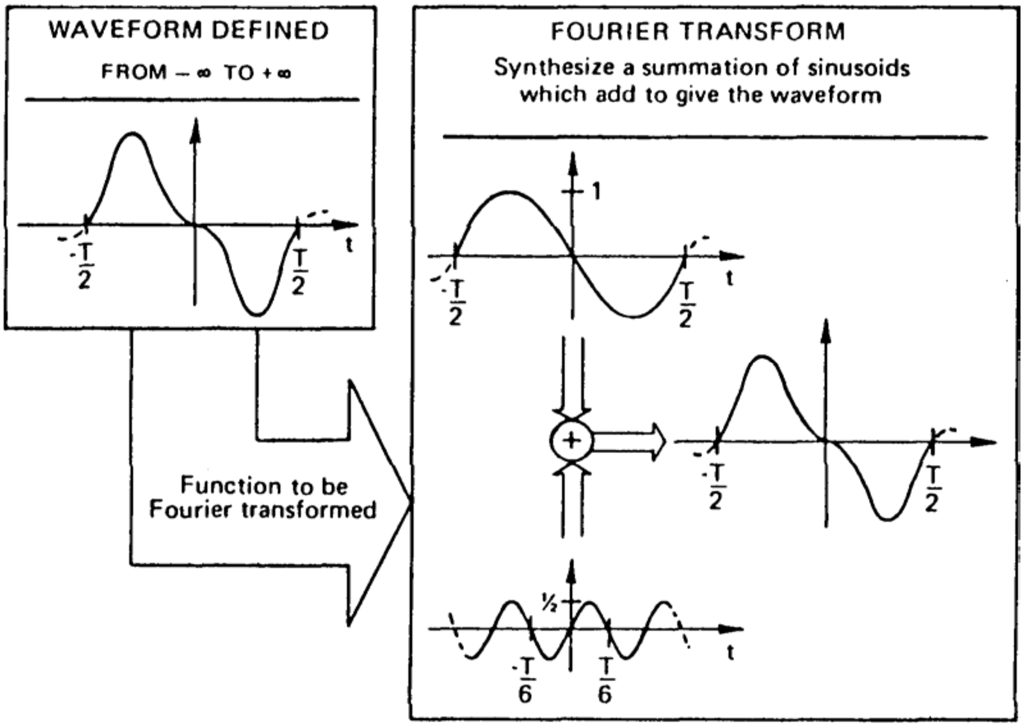
Adding another term refines the shape





And the process continues.

What is important to realize is that if the waveform is composed of a finite number of points, a Fourier series of the same number of points can reproduce that exact waveform.



Analysis Method

1) Decompose error $\{\varepsilon\}^m$ into Fourier components
2) Examine $\{\varepsilon\}^{m+1} = G\{\varepsilon\}^m$ for each component

Example: $\nabla^2 u + fu = g$ (f $\leq$ 0 was given)

ADI formulation:

Step 1: Implicit in X

$$\left(-\omega + \delta_x^2 + f\frac{h^2}{2}\right)u_{i,j}^{m+1} = \left(-\omega - \delta_y^2 - f\frac{h^2}{2}\right)u_{i,j}^{m} + gh^2$$

Setp 2: Implicit in Y

$$\left(-\omega + \delta_y^2 + f\frac{h^2}{2}\right)u_{i,j}^{m+2} = \left(-\omega - \delta_x^2 - f\frac{h^2}{2}\right)u_{i,j}^{m+1} + gh^2$$

Let $\varepsilon^m = u^m - u_{Algebraic}$ (gh^2 will cancel)

$$\left(-\omega + \delta_x^2 + f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+1} = \left(-\omega - \delta_y^2 - f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m}$$

$$\left(-\omega + \delta_y^2 + f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+2} = \left(-\omega - \delta_x^2 - f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+1}$$

Recall that $\delta_x^2 \varepsilon_{i,j} = \varepsilon_{i-1,j} - 2\varepsilon_{i,j} + \varepsilon_{i+1,j}$

1. Harmonic Decomposition

$$\varepsilon^{m}(x,y)=\sum_{k,n} C^{m}_{k,n}\exp(i\sigma_k x)\exp(i\beta_n y)$$

Because of linearity – only consider 1 k,n term.

Each harmonic grows, or decays, separately of other terms. Therefore, the subscripts can be dropped.

$$\sigma=\frac{2\pi}{L_x} \qquad and \qquad \beta=\frac{2\pi}{L_y}$$

$$2h \le L_x \le Kh \to \infty \;\; and \;\; 2h \le L_y \le Nh \to \infty$$

C^m – complex amplitude of waveform which for ADI convergence requires that:

$$\lim_{m\to\infty} C^m_{k,n} = 0 \qquad or \qquad \left|\frac{C^{m+2}}{C^m}\right| < 1$$

Since,

$$\varepsilon^m(x,y) = C^m \exp(i\sigma x)\exp(i\beta y)$$

$$\varepsilon^m(x+h,y) = C^m \exp(i\sigma(x+h))\exp(i\beta y)$$

$$\varepsilon^m(x+h,y) = \exp(i\sigma h)\varepsilon^m(x,y)$$

and

$$\varepsilon^m(x-h,y) = \exp(-i\sigma h)\varepsilon^m(x,y)$$

Recall Trig: $\exp(i\alpha) + \exp(-i\alpha) = 2\cos(\alpha)$

Then:

$$\delta_x^2 \varepsilon_{i,j} = \varepsilon_{i-1,j} - 2\varepsilon_{i,j} + \varepsilon_{i+1,j}$$

$$\delta_x^2 \varepsilon_{i,j} = \exp(-i\sigma h)\varepsilon_{i,j} - 2\varepsilon_{i,j} + \exp(i\sigma h)\varepsilon_{i,j}$$

$$\delta_x^2 \varepsilon_{i,j} = 2[\cos(\sigma h) - 1]\varepsilon_{i,j} \qquad and$$

$$\delta_y^2 \varepsilon_{i,j} = 2[\cos(\beta h) - 1]\varepsilon_{i,j}$$

Returning to the ADI formulation:

$$\left(-\omega+\delta_x^2+f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+1}=\left(-\omega-\delta_y^2-f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m}$$

$$\left(-\omega+\delta_y^2+f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+2}=\left(-\omega-\delta_x^2-f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+1}$$

$$\left(-\omega+2[\cos(\sigma h)-1]+f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+1}=\left(-\omega-2[\cos(\beta h)-1]-f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m}$$

$$\left(-\omega+2[\cos(\beta h)-1]+f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+2}=\left(-\omega-2[\cos(\sigma h)-1]-f\frac{h^2}{2}\right)\varepsilon_{i,j}^{m+1}$$

Rearranging:

$$\frac{\varepsilon_{i,j}^{m+2}}{\varepsilon_{i,j}^{m}}=\frac{C^{m+2}}{C^{m}}=\frac{\dfrac{\left(-\omega-2[\cos(\sigma h)-1]-f\dfrac{h^2}{2}\right)}{\left(-\omega+2[\cos(\beta h)-1]+f\dfrac{h^2}{2}\right)}\varepsilon_{i,j}^{m+1}}{\dfrac{\left(-\omega+2[\cos(\sigma h)-1]+f\dfrac{h^2}{2}\right)}{\left(-\omega-2[\cos(\beta h)-1]-f\dfrac{h^2}{2}\right)}\varepsilon_{i,j}^{m+1}}$$

Note: -2 ≤ [cos(term)-1] ≤ 0 and f ≤ 0

Therefore, define the following terms A and B:

$$A = -2[\cos(\sigma h) - 1] - f\,\frac{h^2}{2} \ge 0$$

and

$$B = -2[\cos(\beta h) - 1] - f\,\frac{h^2}{2} \ge 0$$

$$\frac{\varepsilon_{i,j}^{m+2}}{\varepsilon_{i,j}^{m}} = \frac{C^{m+2}}{C^{m}} = \frac{\dfrac{\left(-\omega + A\right)}{\left(-\omega - B\right)}\,\varepsilon_{i,j}^{m+1}}{\dfrac{\left(-\omega - A\right)}{\left(-\omega + B\right)}\,\varepsilon_{i,j}^{m+1}} = \frac{\left(A-\omega\right)}{-\left(A+\omega\right)} \quad \frac{\left(B-\omega\right)}{-\left(B+\omega\right)}$$

$$for \quad \omega > 0 \qquad \left|\frac{\left(A-\omega\right)}{-\left(A+\omega\right)}\right| \le 1 \quad and \qquad \left|\frac{\left(B-\omega\right)}{-\left(B+\omega\right)}\right| \le 1$$

$\therefore$ $\left|\dfrac{C^{m+2}}{C^{m}}\right| \le 1$ $\quad \omega \ge 0$ Unconditionally Stable

And if: $\omega < 0 \quad then \quad \left|\dfrac{C^{m+2}}{C^{m}}\right| > 1$ Unconditionally Unstable